arXiv:1502.03309v3 [math.CA] 15 Feb 2015

On the integral representations for Dunkl kernels of type A_2.

Béchir Amri

Université Tunis El Manar
Faculté des Sciences de Tunis
Département de Mathématiques
Laboratoire d'Analyse Mathématiques et Applications LR11ES11
2092- El Manar I, Tunis TUNISIA
bechir.amri@ipeit.rnu.tn

Abstract

We give an explicit integral formula for the Dunkl kernel associated to root system of type A_2 and parameter $k > 0$, by exploiting recent result in [1]. [1]

1 Introduction

In this paper we mainly focus on Dunkl kernels associated to root systems of type A, for a purpose of finding an explicit representation integrals for these functions, following our recent work on symmetric case. We outline here a simple method that leads us to such formulas for the A_2 root system and provide a short and elementary proof of Dunkl's formula for the intertwining operator established in [2] for parameter $k > 1/2$. General references are [2, 3, 4, 5, 7, 8, 9].

Following the notations given in [1], letting $\mathbb{V}$ be the hyperplane,

$$\mathbb{V} = \{(x, y, z) \in \mathbb{R}^3;\ x + y + z = 0\}$$

and the root system $R = \{\pm(e_1 - e_2),\ \pm(e_1 - e_3),\ \pm(e_2 - e_3)\}$ where (e_1, e_2, e_3) is the standard basis of the Euclidean space $\mathbb{R}^3$. Fixe $(e_1 - e_2, e_2 - e_3)$ as the basis of simple root and C the corresponding fundamental Weyl chamber,

$$C = \{\lambda = (\lambda_1, \lambda_2\lambda_3); \quad \lambda_3 < \lambda_2 < \lambda_1\}.$$

The Weyl group is isomorphic to the symmetric group S_3. The Dunkl operators are given by

$$T_i = \frac{\partial}{\partial x_i} + k \sum_{1\leq j\neq i\leq 3} \frac{1 - s_{i,j}}{x_i - x_j}, \qquad i = 1, 2, 3$$

where k is a positive real parameter and $s_{i,j}$ acts on functions of vaiables (x_1, x_2, x_3) by interchanging the variables x_i and x_j. The Dunkl kernel $E_k(., y)$, $y \in \mathbb{R}^3$, characterized by being the unique solution of the following eigenvalue problem

$$T_i(E_k(., y))(x) = y_i E_k(x, y); \qquad E(0, y) = 0, \quad i = 1, 2, 3.$$

[1]

Key words and phrases: Dunkl operators, root systems, reflection groups.
2010 Mathematics Subject Classification: Primary 33E30; Secondary 17B22,20F55.

Let J_k the generalized Bessel function associated with R and k, given by

$$J_k(x,y) = \frac{1}{6}\sum_{\sigma\in G} E_k(\sigma.x,y). \tag{1.1}$$

The functions J_k are related to the ordinary modified Bessel functions $\mathcal{J}_{k-\frac{1}{2}}$ by (see [1]):

$$\begin{aligned} J_k(\mu,\lambda) &= \frac{\Gamma(3k)}{V(\lambda)^{2k-1}\Gamma(k)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}) \\ &\qquad (\nu_1-\nu_2)W_k(\mu,\lambda)d\nu_1 d\nu_2, \end{aligned}$$

for all $\lambda = (\lambda_1,\lambda_2,\lambda_3) \in \mathbb{V}\cap C$ and $\mu\in\mathbb{R}^3$, where

$$\begin{aligned} V(\lambda) &= (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_3) \\ W_k(\nu,\lambda) &= \Big((\lambda_1-\nu_1)(\lambda_1-\nu_2)(\lambda_2-\nu_2)(\nu_1-\lambda_2)(\nu_1-\lambda_3)(\nu_2-\lambda_3)\Big)^{k-1}. \end{aligned}$$

Recall here that

$$\mathcal{J}_{k-\frac{1}{2}}(z) = \frac{\Gamma(2k)}{2^{2k-1}\Gamma(k)^2}\int_{-1}^{1} e^{zt}(1-t^2)^{k-1}dt; \qquad z\in\mathbb{R}.$$

In the next section we shall use this fact to construct an integral formula for E_k. The following theorem is the main result of this article.

Theorem 1. *The Dunkl kernel of type A_2 has the following integral formula*

$$\begin{aligned} E_k(\mu,\lambda) &= \frac{\Gamma(3k)}{V(\lambda)^{2k}\Gamma(k)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2}\Big\{3(\lambda_1-\lambda_2)(\nu_1-\nu_2)\mathcal{J}_{k-\frac{1}{2}}\Big(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\Big) \\ &\quad -6\Big(\nu_1\nu_2+\frac{\lambda_3}{2}(\nu_1+\nu_2)+\lambda_1\lambda_2\Big)\mathcal{J}'_{k-\frac{1}{2}}\Big(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\Big)\Big\} \\ &\quad (\lambda_3-\nu_1)(\lambda_3-\nu_2)e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}W_k(\nu,\lambda)d\nu_1 d\nu_2, \end{aligned} \tag{1.2}$$

for all $\lambda\in\mathbb{V}\cap C$ and $\mu\in\mathbb{R}^3$.

2 Outline the proof

An interesting relation between J_k and J_{k+1} is given in ([6], p.369) by the following functional equation

$$T_V(J_{k+1}(.,y)V(.))(x) = \gamma_k J_k(x,y) \tag{2.1}$$

where $T_V = (T_1-T_2)(T_2-T_3(T_1-T_3))$ and $\gamma_k = T_V(V(.))(0) = \Big((2k+1)(3k+1)(3k+2)\Big)^{-1}$. This together with Proposition 1.4 of [4] implies

$$\sum_{\sigma\in G} det(\sigma)E_k(\sigma.\mu,\lambda) = \gamma_k V(\mu)V(\lambda)J_{k+1}(\mu,\lambda). \tag{2.2}$$

Combining (2.2) with (1.1) yields for all $\mu\in\mathbb{R}^3$ and $\lambda\in\mathbb{V}$

$$E_k(\mu,\lambda)+E_k(\mu,\sigma.\lambda)+E_k(\mu,\sigma^2.\lambda) = \frac{1}{2}\Big(\gamma_k V(\lambda)V(\mu)J_{k+1}(\mu,\lambda)+6J_k(\mu,\lambda)\Big) \tag{2.3}$$

where $\sigma = s_{1,3}s_{1,2}$. This is a starting point from which we have the following

Lemma 1. *Let $\lambda \in \mathbb{V}$ and T be the operator*

$$T = \frac{2\lambda_1 + \lambda_2}{\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2} T_1 + \frac{2\lambda_2 + \lambda_1}{\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2} T_2 + 1 = \alpha(\lambda) T_1 + \beta(\lambda) T_2 + 1$$

Then we have

$$E_k(\mu, \lambda) = T\Big(\frac{\gamma_k}{6}\, V(\lambda)\, V(.) J_{k+1}(., \lambda) + J_k(., \lambda)\Big)(\mu), \qquad \mu \in \mathbb{R}^3.$$

The proof is a straightforward calculation which we shall omit. However, to obtain our integral formula for E_k, it therefore comes down to express the following terms with suitable integrals

(i) $V(\mu) J_{k+1}(\mu, \lambda)$

(ii) $(\mu_1 - \mu_2)(\mu_2 - \mu_3) J_{k+1}(\mu, \lambda)$

(iii) $(\mu_1 - \mu_2)(\mu_1 - \mu_3) J_{k+1}(\mu, \lambda)$

(iv) $T_1(V(.) J_{k+1}(., \lambda)(\mu) = V(\mu) \dfrac{\partial J_{k+1}}{\partial \mu_1}(\mu, \lambda) + (2k+1) \dfrac{\partial V(\mu)}{\partial \mu_1} J_{k+1}(\mu, \lambda)$

(v) $T_2(V(.) J_{k+1}(., \lambda)(\mu) = V(\mu) \dfrac{\partial J_{k+1}}{\partial \mu_2}(\mu, \lambda) + (2k+1) \dfrac{\partial V(\mu)}{\partial \mu_2} J_{k+1}(\mu, \lambda)$

We will need to use the following classical equations of the modified Bessel function $\mathcal{J}_\alpha$, $\alpha > -\frac{1}{2}$,

$$z \mathcal{J}_{\alpha+1}(z) = 2(\alpha+1) \mathcal{J}'_\alpha(z) \tag{2.4}$$

$$\mathcal{J}_\alpha(z) = \mathcal{J}''_\alpha(z) + \frac{2\alpha+1}{z} \mathcal{J}'_\alpha(z) \tag{2.5}$$

and the following facts:

$$(\mu_1 - \mu_2)(\mu_1 - \mu_3) = \frac{(\mu_1 - \mu_2)(\mu_1 + \mu_2 - 2\mu_3) + (\mu_1 - \mu_2)^2}{2} \tag{2.6}$$

$$(\mu_1 - \mu_2)(\mu_2 - \mu_3) = \frac{(\mu_1 - \mu_2)(\mu_1 + \mu_2 - 2\mu_3) - (\mu_1 - \mu_2)^2}{2} \tag{2.7}$$

$$(\mu_1 - \mu_3)(\mu_2 - \mu_3) = \frac{(\mu_1 + \mu_2 - 2\mu_3)^2 - (\mu_1 - \mu_2)^2}{4} \tag{2.8}$$

$$V(\mu) = \frac{(\mu_1 + \mu_2 - 2\mu_3)^2(\mu_1 - \mu_2) - (\mu_1 - \mu_2)^3}{4}. \tag{2.9}$$

First, from (2.4) we have

$$\begin{aligned}(\mu_1 - \mu_2) & J_{k+1}(\mu, \lambda) \\ &= \frac{(4k+2)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}} \mathcal{J}'_{k-\frac{1}{2}}\Big(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2}\Big) \\ &\qquad W_{k+1}(\nu, \lambda)\, d\nu_1 d\nu_2\end{aligned}$$

and by using integration by parts,

$$\begin{aligned}(\mu_1 - \mu_2)^2 & J_{k+1}(\mu, \lambda) \\ &= \frac{(4k+2)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3} \int_{\lambda_2}^{\lambda_1} \int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}} \mathcal{J}_{k-\frac{1}{2}}\Big(\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2)}{2}\Big) \\ &\qquad (\partial_{\nu_1} - \partial_{\nu_2}) W_{k+1}(\nu, \lambda)\, d\nu_1 d\nu_2.\end{aligned}$$

Making use of (2.5) we have

$(\mu_1-\mu_2)^3 J_{k+1}(\mu,\lambda)$

$$
\begin{aligned}
= & -\frac{(4k+2)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2}(\mu_1-\mu_2)e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}''_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right) \\
& \qquad (\partial_{\nu_1}-\partial_{\nu_2})W_{k+1}(\nu,\lambda)\, d\nu_1 d\nu_2 \\
& -\frac{4k(4k+2)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2}e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right) \\
& \qquad \frac{(\partial_{\nu_1}-\partial_{\nu_2})W_{k+1}(\nu,\lambda)}{\nu_1-\nu_2}\, d\nu_1 d\nu_2.
\end{aligned}
$$

and by integration by parts one-time,

$(\mu_1+\mu_2-2\mu_3)^2(\mu_1-\mu_2)J_{k+1}(\mu,\lambda)$

$$
\begin{aligned}
= & -\frac{(4k+2)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2}(\mu_1+\mu_2-2\mu_3)e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right) \\
& \qquad (\partial_{\nu_1}+\partial_{\nu_2})W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2.
\end{aligned}
$$

Note that the condition $k>0$ is not sufficient to make an integration by parts again using the derivative operators $\partial_{\nu_1}+\partial_{\nu_2}$ or $\partial_{\nu_1}-\partial_{\nu_2}$, because the appearance of $\partial^2_{\nu_1}W_{k+1}$ and $\partial^2_{\nu_2}W_{k+1}$. However, we see that

$$
\begin{aligned}
- & (\mu_1+\mu_2-2\mu_3)e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)(\partial_{\nu_1}+\partial_{\nu_2})W_{k+1}(\nu,\lambda) \\
+ & (\mu_1-\mu_2)e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}''_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)(\partial_{\nu_1}-\partial_{\nu_2})W_{k+1}(\nu,\lambda) \\
= & -2\partial_{\nu_1}\left\{e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\right\}\partial_{\nu_2}W_{k+1}(\nu,\lambda) \\
& -2\partial_{\nu_2}\left\{e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\right\}\partial_{\nu_1}W_{k+1}(\nu,\lambda).
\end{aligned}
$$

Thus from (2.9) and integration by parts we obtain

$V(\mu)J_{k+1}(\mu,\lambda)$

$$
\begin{aligned}
= & \frac{(4k+2)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2}e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right) \\
& \qquad \left(\partial_{\nu_1}\partial_{\nu_2}+k\frac{\partial_{\nu_1}-\partial_{\nu_2}}{\nu_1-\nu_2}\right)W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2
\end{aligned}
$$

which is a nice integral formula for (i).

Next, using (2.6)and (2.7) with integration by parts,

$$
\begin{aligned}
&(\mu_1-\mu_2)(\mu_1-\mu_3)J_{k+1}(\mu,\lambda)\\
&\quad = -\frac{(4k+2)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\qquad (\partial_{\nu_1}-\partial_{\nu_2})W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\\
&\qquad -\frac{(4k+2)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\qquad (\partial_{\nu_1}+\partial_{\nu_2})W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mu_1-\mu_2)(\mu_2-\mu_3)J_{k+1}(\mu,\lambda)\\
&\quad = \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\qquad (\partial_{\nu_1}-\partial_{\nu_2})W_{k+1}(\lambda,\mu)d\nu_1 d\nu_2\\
&\quad -\frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2})\\
&\qquad\qquad (\partial_{\nu_1}+\partial_{\nu_2})W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2.
\end{aligned}
$$

For (iv) we make use of the fact that

$$
z\mathcal{J}'_{\alpha+1}(z) = 2(\alpha+1)\Big(\mathcal{J}_\alpha(z)-\mathcal{J}_{\alpha+1}(z)\Big),
$$

and write

$$
\begin{aligned}
&V(\mu)\frac{\partial J_{k+1}}{\partial \mu_1}(\mu)\\
&= \frac{\Gamma(3k+3)}{2V(\lambda)^{2k+1}\Gamma(k+1)^3}V(\mu)\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k+\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\qquad (\nu_1-\nu_2)(\nu_1+\nu_2)W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\\
&+\frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}(\mu_1-\mu_3)(\mu_2-\mu_3)\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\qquad (\nu_1-\nu_2)W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\\
&-(2k+1)(\mu_1-\mu_3)(\mu_2-\mu_3)J_{k+1}\ .
\end{aligned}
$$

Proceeding as for the integral representation of (i), we have

$$
\begin{aligned}
&\frac{\Gamma(3k+3)}{2V(\lambda)^{2k+1}\Gamma(k+1)^3}\Big\{V(\mu)\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k+\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\qquad (\nu_1-\nu_2)(\nu_1+\nu_2)W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\Big\}\\
&= \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad \left\{\partial_{\nu_1}\partial_{\nu_2}\Big((\nu_1+\nu_2)W_{k+1}(\nu,\lambda)\Big)+k\ \frac{(\partial_{\nu_1}-\partial_{\nu_2})\Big((\nu_1+\nu_2)W_{k+1}(\nu,\lambda)\Big)}{\nu_1-\nu_2}\right\}d\nu_1 d\nu_2\ .
\end{aligned}
$$

On the other hand, by using (2.5) and (2.8) with integration by parts,

$$
\begin{aligned}
&\frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}(\mu_1-\mu_3)(\mu_2-\mu_3)\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\qquad (\nu_1-\nu_2)W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\\
=\ & -\frac{(2k+1)\Gamma(3k+3)}{4V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2}(\mu_1+\mu_2-2\mu_3)e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\qquad (\partial\nu_1+\partial\nu_2)\Big(\nu_1-\nu_2\Big)W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\\
+\ & \frac{(2k+1)\Gamma(3k+3)}{4V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2}(\mu_1-\mu_2)e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\qquad (\partial\nu_1-\partial\nu_2)\Big(\nu_1-\nu_2)W_{k+1}(\nu,\lambda)\Big)d\nu_1 d\nu_2\\
+\ & \frac{k(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\qquad (\partial\nu_1-\partial\nu_2)W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2.
\end{aligned}
$$

As we noted above for the use of integration by parts a second time, we can do it by the following observations

$$
\begin{aligned}
-\ & (\mu_1+\mu_2-2\mu_3)e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)(\partial_{\nu_1}+\partial_{\nu_2})\Big((\nu_1-\nu_2)W_{k+1}(\nu,\lambda)\Big)\\
+\ & (\mu_1-\mu_2)e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)(\partial_{\nu_1}-\partial_{\nu_2})\Big((\nu_1-\nu_2)W_{k+1}(\nu,\lambda)\Big)\\
=\ & -2\partial_{\nu_1}\left\{e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\right\}\partial_{\nu_2}\Big((\nu_1-\nu_2)W_{k+1}(\nu,\lambda)\Big)\\
& -2\partial_{\nu_2}\left\{e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\right\}\partial_{\nu_1}\Big((\nu_1-\nu_2)W_{k+1}(\nu,\lambda)\Big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}(\mu_1-\mu_3)(\mu_2-\mu_3)\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\qquad (\nu_1-\nu_2)W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\\
=\ & \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2})\\
&\qquad\qquad \Big\{\partial\nu_1\partial\nu_2\Big((\nu_1-\nu_2)W_{k+1}(\nu,\lambda)\Big)+k(\partial\nu_1-\partial\nu_2)W_{k+1}(\nu,\lambda)\Big\}\,d\nu_1 d\nu_2\ .
\end{aligned}
$$

From these calculations it follows that

$$
\begin{aligned}
&T_1(V(.)J_{k+1}(.,\lambda))(\mu)\\
&= V(\mu)\frac{\partial J_{k+1}}{\partial \mu_1}(\mu) + (2k+1)\Big((\mu_1-\mu_3)(\mu_2-\mu_3)+(\mu_1-\mu_2)(\mu_2-\mu_3)\Big)J_{k+1}(\mu)\\
&= \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\left\{(\nu_1+\nu_2)\left(\partial\nu_1\partial\nu_2 + k\frac{\partial_{\nu_1}-\partial_{\nu_2}}{\nu_1-\nu_2}\right)-2k(\partial_{\nu_1}+\partial_{\nu_2})\right\}W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\\
&+\quad \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad(\nu_1-\nu_2)\left(\partial\nu_1\partial\nu_2+3k\frac{(\partial_{\nu_1}-\partial_{\nu_2})}{\nu_1-\nu_2}\right)W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2
\end{aligned}
$$

By the fact that

$$T_2(V(.)J_{k+1}(.,\lambda))(\mu_1,\mu_2,\mu_3) = -T_1(V(.)J_{k+1}(.,\lambda))(\mu_2,\mu_1,\mu_3)$$

we also have

$$
\begin{aligned}
&T_2(V(.)J_{k+1}(.,\lambda))(\mu)\\
&= \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\qquad\left\{(\nu_1+\nu_2)\left(\partial\nu_1\partial\nu_2 + k\frac{\partial_{\nu_1}-\partial_{\nu_2}}{\nu_1-\nu_2}\right)-2k(\partial_{\nu_1}+\partial_{\nu_2})\right\}W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\\
&-\quad \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)(\nu_1-\nu_2)\\
&\qquad\left(\partial\nu_1\partial\nu_2+3k\frac{(\partial_{\nu_1}-\partial_{\nu_2})}{\nu_1-\nu_2}\right)W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\ .
\end{aligned}
$$

By virtue of these integral formulas we obtain

$$
\begin{aligned}
&T(V(.)J_{k+1}(.,\lambda))(\mu)\\
&= \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)\\
&\left\{((\alpha+\beta)(\nu_1+\nu_2)+2)\left(\partial\nu_1\partial\nu_2 + k\frac{\partial_{\nu_1}-\partial_{\nu_2}}{\nu_1-\nu_2}\right)-2k(\alpha+\beta)(\partial_{\nu_1}+\partial_{\nu_2})\right\}W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\\
&+\quad \frac{(2k+1)\Gamma(3k+3)}{V(\lambda)^{2k+1}\Gamma(k+1)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)(\nu_1-\nu_2)\\
&\qquad(\alpha-\beta)\left(\partial\nu_1\partial\nu_2+3k\frac{\partial_{\nu_1}-\partial_{\nu_2}}{\nu_1-\nu_2}\right)W_{k+1}(\nu,\lambda)d\nu_1 d\nu_2\ .
\end{aligned}
$$

Put $a(\lambda) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ and $b(\lambda) = -\lambda_1\lambda_2\lambda_3$, we have

$$\left(\partial\nu_1\partial\nu_2 + k\frac{\partial_{\nu_1}-\partial_{\nu_2}}{\nu_1-\nu_2}\right)W_{k+1}(\nu,\lambda) = -k^2\Big(6\nu_1^2\nu_2^2 + 2a(\nu_1^2+\nu_2^2+\nu_1\nu_2) + 3b(\nu_1+\nu_2)\Big)W_k(\nu,\lambda)$$

$$\left\{(\nu_1+\nu_2)\left(\partial\nu_1\partial\nu_2 + k\frac{\partial_{\nu_1}-\partial_{\nu_2}}{\nu_1-\nu_2}\right) - 2k(\partial_{\nu_1}+\partial_{\nu_2})\right\}W_{k+1}(\nu,\lambda)$$
$$= k^2\Big(2a\nu_1\nu_2(\nu_1+\nu_2) + 3b(\nu_1-\nu_2)^2 + 2a^2(\nu_1+\nu_2) + 4ab\Big)W_k(\nu,\lambda)$$

$$\left(\partial\nu_1\partial\nu_2 + 3k\frac{(\partial_{\nu_1}-\partial_{\nu_2})}{\nu_1-\nu_2}\right)W_{k+1}(\nu,\lambda) = k^2\Big(-6a\nu_1\nu_2 - 9b(\nu_1+\nu_2) + 2a^2\Big)W_k(\nu,\lambda)$$

$$\left\{((\alpha+\beta)(\nu_1+\nu_2)+2)\left(\partial\nu_1\partial\nu_2 + k\frac{\partial_{\nu_1}-\partial_{\nu_2}}{\nu_1-\nu_2}\right) - 2k(\alpha+\beta)(\partial_{\nu_1}+\partial_{\nu_2})\right\}W_{k+1}(\nu,\lambda)$$
$$= -k^2\Big(12\nu_1^2\nu_2^2 + 4a(\nu_1^2+\nu_2^2+\nu_1\nu_2) + 6b(\nu_1+\nu_2)\Big)W_k(\nu,\lambda)$$
$$+(\alpha+\beta)k^2\Big(2a\nu_1\nu_2(\nu_1+\nu_2) + 3b(\nu_1-\nu_2)^2 + 2a^2(\nu_1+\nu_2) + 4ab\Big)W_k(\nu,\lambda).$$

We finally obtain

$$\begin{aligned}
E_k(\mu,\lambda) &= \frac{\Gamma(3k)}{V(\lambda)^{2k}\Gamma(k)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}\left(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\right)(\nu_1-\nu_2) \\
&\quad \left\{\frac{\alpha-\beta}{2}\Big(-6a\nu_1\nu_2 - 9b(\nu_1+\nu_2) + 2a^2\Big) + \Big(\frac{\alpha+\beta}{2}(\nu_1+\nu_2)+1\Big)V(\lambda)\right\}W_k(\nu,\lambda)d\nu_1 d\nu_2 \\
&+ \frac{\Gamma(3k)}{V(\lambda)^{2k}\Gamma(k)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}'_{k-\frac{1}{2}}\big(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2}\big) \\
&\quad \Big\{\frac{(\alpha+\beta)}{2}\Big(2a\nu_1\nu_2(\nu_1+\nu_2) + 3b(\nu_1-\nu_2)^2 + 2a^2(\nu_1+\nu_2) + 4ab\Big) \\
&- \Big(6\nu_1^2\nu_2^2 + 2a(\nu_1^2+\nu_2^2+\nu_1\nu_2) + 3b(\nu_1+\nu_2)\Big) + \frac{\alpha-\beta}{2}(\nu_1-\nu_2)^2V(\lambda)\Big\}W_k(\nu,\lambda)d\nu_1 d\nu_2
\end{aligned}$$

where,

$$\begin{aligned}
&\frac{\alpha-\beta}{2}\Big(-6a\nu_1\nu_2 - 9b(\nu_1+\nu_2) + 2a^2\Big) + \Big(\frac{\alpha+\beta}{2}(\nu_1+\nu_2)+1\Big)V(\lambda) \\
&= 3(\lambda_1-\lambda_2)\nu_1\nu_2 + 3(\lambda_1^2-\lambda_2^2)(\nu_1+\nu_2) + 3\lambda_3^2(\lambda_1-\lambda_2) \\
&= 3(\lambda_1-\lambda_2)(\lambda_3-\nu_1)(\lambda_3-\nu_2),
\end{aligned}$$

$$\begin{aligned}
&\Big\{\frac{(\alpha+\beta)}{2}\Big(2a\nu_1\nu_2(\nu_1+\nu_2) + 3b(\nu_1-\nu_2)^2 + 2a^2(\nu_1+\nu_2) + 4ab\Big) \\
-&\Big(6\nu_1^2\nu_2^2 + 2a(\nu_1^2+\nu_2^2+\nu_1\nu_2) + 3b(\nu_1+\nu_2)\Big) + \frac{\alpha-\beta}{2}(\nu_1-\nu_2)^2V(\lambda)\Big\} \\
=& -6\nu_1^2\nu_2^2 + 3\lambda_3\nu_1\nu_2(\nu_1+\nu_2) - 3\lambda_3(\lambda_1^2+\lambda_2^2)(\nu_1+\nu_2) - 6\lambda_1\lambda_2\lambda_3^2 - 2(\nu_1^2+\nu_2^2+\nu_1\nu_2)(\lambda_1\lambda_2-\lambda_3^2) \\
&+(2\lambda_1\lambda_2+\lambda_3^2)(\nu_1-\nu_2)^2 \\
=& -6(\lambda_3-\nu_1)(\lambda_3-\nu_2)\Big(\nu_1\nu_2 + \frac{\lambda_3}{2}(\nu_1+\nu_2) + \lambda_1\lambda_2\Big).
\end{aligned}$$

This conclude the proof of Theorem the main result.

Now if we equippped the space $\mathbb{V}$ with the basis ($e_1-e_3,\ e_2-e_3$) and with the Lebesgue measure $d\nu = d\nu_1 d\nu_2$, then we can state

Corollary 1. *The Dunkl kernel E_k connected with the exponential function by*

$$E_k(\mu,\lambda) = \int_{co(\lambda)} e^{\langle \mu,\nu\rangle} F_k\left(\frac{\nu_1+\nu_2}{2}, \frac{\nu_1-\nu_2}{2}, \lambda\right)\, d\nu \tag{2.10}$$

where $co(\lambda) = \{\nu \in \mathbb{V},\ \lambda_3 \le\ \nu_1,\ \nu_2,\ \nu_3\ \le \lambda_1\}$, the convex hull of the orbit $G.\lambda$ and the function F_k is given by

$$\begin{aligned} &F_k(x,y,\lambda) = \\ &\frac{\Gamma(2k)\Gamma(3k)}{2^{2k-2}\Gamma(k)^5 V(\lambda)^{2k}} \int_{\max(|y|,|x-\lambda_2|)}^{\min(x-\lambda_3,\lambda_1-x)} \Big(3z^2(2y+\lambda_1-\lambda_2) - 6y(x-\lambda_1)(x-\lambda_2)\Big) \\ &\qquad \left(\frac{(\lambda_3-x)^2-z^2}{z^2}\right)^k \Big((z^2-y^2)((\lambda_1-x)^2-z^2)(z^2-(\lambda_2-x)^2)\Big)^{k-1} dz, \end{aligned}$$

if $\max(|y|,|x-\lambda_2|) \le \min(x-\lambda_3,\lambda_1-x)$ *and equal* 0 *otherwise.*

Proof. Recall that

$$\begin{aligned} \mathcal{J}_{k-\frac{1}{2}}((\mu_1-\mu_2)z) &= \frac{\Gamma(2k)}{2^{2k-1}\Gamma(k)^2} \int_{\mathbb{R}} e^{(\mu_1-\mu_2)y}(1-\frac{y^2}{z^2})^{k-1}\chi_{[-1,1]}(\frac{y}{z})z^{-1}dy, \\ \mathcal{J}'_{k-\frac{1}{2}}((\mu_1-\mu_2)z) &= \frac{\Gamma(2k)}{2^{2k-1}\Gamma(k)^2} \int_{\mathbb{R}} e^{(\mu_1-\mu_2)y}(1-\frac{y^2}{z^2})^{k-1}\frac{y}{z^2}\chi_{[-1,1]}(\frac{y}{z})dy. \end{aligned}$$

Inserting these into (1.2) and making use the change of variables:

$$x = \frac{\nu_1+\nu_2}{2}, \quad z = \frac{\nu_1-\nu_2}{2},$$

with Fubuni's Theorem, we obtain

$$E_k(\mu,\lambda) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{(\mu_1+\mu_2-2\mu_3)x+(\mu_1-\mu_2)y} F_k(x,y,\lambda)dxdy \tag{2.11}$$

where

$$\begin{aligned} &F_k(x,y,\lambda) = \\ &\frac{\Gamma(2k)\Gamma(3k)}{2^{2k-2}\Gamma(k)^5 V(\lambda)^{2k}} \int_{\mathbb{R}} \Big(3z^2(\lambda_1-\lambda_2) - 6y(x^2-z^2+\lambda_3 x+\lambda_1\lambda_2)\Big) \left(\frac{(\lambda_3-x)^2-z^2}{z^2}\right)^k \\ &\Big((z^2-y^2)(\lambda_1-x)^2-z^2)(z^2-(\lambda_2-x)^2)\Big)^{k-1} \\ &\qquad\qquad \chi_{[-1,1]}(\frac{y}{z})\chi_{[\lambda_1,\lambda_2]}(x+z)\chi_{[\lambda_3,\lambda_2]}(x-z)dz \\ = \quad &\frac{\Gamma(2k)\Gamma(3k)}{2^{2k-2}\Gamma(k)^5 V(\lambda)^{2k}} \int_{\max(|y|,|x-\lambda_2|)}^{\min(x-\lambda_3,\lambda_1-x)} \Big(3z^2(2y+\lambda_1-\lambda_2) - 6y(x-\lambda_1)(x-\lambda_2)\Big) \\ &\qquad \left(\frac{(\lambda_3-x)^2-z^2}{z^2}\right)^k \Big((z^2-y^2)((\lambda_1-x)^2-z^2)(z^2-(\lambda_2-x)^2)\Big)^{k-1} dz \end{aligned}$$

where we used the fact that

$$\chi_{[-1,1]}\left(\frac{y}{z}\right)\chi_{[\lambda_1,\lambda_2]}(x+z)\chi_{[\lambda_3,\lambda_2]}(x-z) = \chi_{\max(|y|,|x-\lambda_2|)\le z\le \min(x-\lambda_3,\lambda_1-x)}.$$

Now, the change of variables

$$x = \frac{\nu_1 + \nu_2}{2}, \quad y = \frac{\nu_1 - \nu_2}{2},$$

gives

$$E_k(\mu, \lambda) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\langle \mu, \nu \rangle} F_k \left(\frac{\nu_1 + \nu_2}{2}, \frac{\nu_1 - \nu_2}{2}, \lambda \right) d\nu_1 \nu_2. \tag{2.12}$$

To achieve the proof we use that

$$\left\{ \nu \in \mathbb{V}; \quad \max\left(\frac{|\nu_1 - \nu_2|}{2}, \left| \frac{\nu_1 + \nu_2}{2} - \lambda_2 \right| \right) \leq \min\left(\frac{\nu_1 + \nu_2}{2} - \lambda_3, \lambda_1 - \frac{\nu_1 + \nu_2}{2} \right) \right\} = co(\lambda).$$

□

References

[1] B. Amri, Note on Bessel functions of type A_{N-1}, Integral Transforms and Special Functions, 25(2014), no.6, 448–461.

[2] *C. F. Dunkl* , Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc., 311 (1989), no. 1, 167183.

[3] C. F. Dunkl, Integral kernels with reflection group invariance, Canadian J. Math. 43 (1991), 1213–1227.

[4] C. F. Dunkl, Intertwining operators associated to the group S_3, Trans. Amer. Math. Soc.,347 (1995), no. 9, 3347–3374.

[5] M. F. E. de Jeu, The Dunkl transform, Invent. Math. 113 (1993), 147–162.

[6] E. M. Opdam, Dunkl operators, Bessel functions, and the discriminant of a finite Coxeter group, Compos. Math. 85 (1993), 333-373.

[7] M. Rösler, Positivity of Dunkl's intertwining operator, Duke Math. J. 98 (1999), 445–463.

[8] M. Rösler, A positive radial product formula for the Dunkl kernel, Trans. Amer. Math. Soc. 355 (2003), 2413–2438.

[9] M. Rösler, *Dunkl operators: theory and applications*. Lecture Notes in Math., 1817, Orthogonal polynomials and special functions, Leuven, 2002, (Springer, Berlin, 2003) 93–135.